

# Joint analysis of bipartite networks collection

JdS 2025

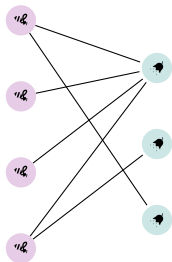
Louis Lacoste, Pierre Barbillon and Sophie Donnet

Laboratoire MIA Paris-Saclay



May 22, 2025

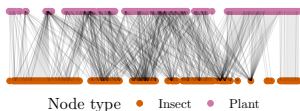
# Why a network?



$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Associated  
adjacency  
matrix

Figure 1: Example of a network



Node type ● Insect ● Plant

- Modeling of various interactions, here ecosystems
- Structure necessary for: biodiversity monitoring, robustness, risk of collapse
- Increasingly available

Figure 2: Plant-pollinator network from Bristol  
Baldock et al., 2019

# Analysis methods for a network

TODO (Supprimable) Several methods :

- Metrics at
  - ▶ node level: degree, centrality...
  - ▶ network level: density, nestedness...
- Node embedding and/or clustering with latent variable models  
Hoff et al., 2002; Snijders and Nowicki, 1997
- Node or network embedding with Graph Convolutional Networks  
Kipf and Welling, 2016

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# Latent Block Model (LBM<sup>1</sup>)

Govaert and Nadif, 2005.

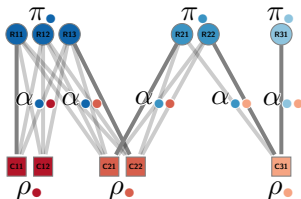


Figure 3: Example of LBM<sup>1</sup>

## Hierarchical model

$$\forall q \in \llbracket 1, Q_1 \rrbracket, \mathbb{P}(Z_i = q) = \pi_q$$

$$\forall r \in \llbracket 1, Q_2 \rrbracket, \mathbb{P}(W_j = r) = \rho_r$$

$$Y_{ij} | Z_i, W_j \sim \mathcal{F}(\alpha_{Z_i, W_j})$$

where

$$|\pi| = Q_1, |\rho| = Q_2, |\alpha| = Q_1 \times Q_2$$

## Concise LBM formula

$$Y \sim \mathcal{F}\text{-BiSBM}_{n_1, n_2}(Q_1, Q_2, \pi, \rho, \alpha)$$

<sup>1</sup>Which I will henceforth call BiSBM

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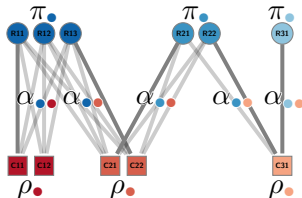


Figure 3: Example of LBM<sup>1</sup>

With

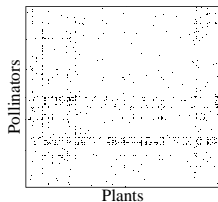
- $Q_1 = |\{\bullet, \bullet, \bullet\}|$  fixed row blocks
- $Q_2 = |\{\bullet, \bullet, \bullet\}|$  fixed column blocks

## Parameters

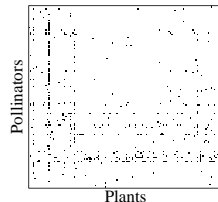
- $\pi_{\bullet} = \mathbb{P}(Z_i = \bullet)$
- $\rho_{\bullet} = \mathbb{P}(W_j = \bullet)$
- $\alpha_{\bullet\bullet} = \mathbb{P}(Y_{ij} = 1 | Z_i = \bullet, W_j = \bullet)$

<sup>1</sup>Which I will henceforth call BiSBM

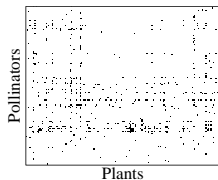
# Multiple networks



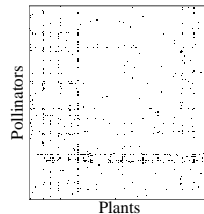
(a) Bristol



(b) Edinburgh



(c) Leeds



(d) Reading

Figure 4: Adjacency matrices, Baldock et al., 2019

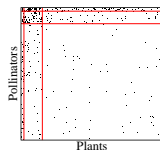
## Bipartite collections separate BiSBM

$$\forall m \in \{1 \dots M\}, Y^m \stackrel{ind}{\sim} \mathcal{F}\text{-BiSBM}_{n_1^m, n_2^m}(Q_1^m, Q_2^m, \pi^m, \rho^m, \alpha^m)$$

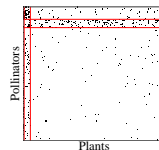


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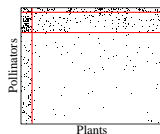
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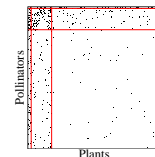
(a) Bristol



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(c) Leeds



(d) Reading

Figure 5: Reordered adjacency matrices, using LBM

# Several joint models

## *iid*-colBiSBM

$$\forall m \in \{1 \dots M\}, Y^m \stackrel{iid}{\sim} \mathcal{F}\text{-BiSBM}_{n_1^m, n_2^m}(Q_1, Q_2, \pi, \rho, \alpha)$$

with  $\theta = (\pi, \rho, \alpha)$ .

## Several joint models

### *iid*-colBiSBM

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### $\pi\rho$ -colBiSBM

$$\forall m \in \{1 \dots M\}, Y^m \stackrel{ind}{\sim} \mathcal{F}\text{-BiSBM}_{n_1^m, n_2^m}(Q_1, Q_2, \pi^m, \rho^m, \alpha)$$

with  $\theta = ((\pi^m)_{m=1, \dots, M}, (\rho^m)_{m=1, \dots, M}, \alpha)$ .

And intermediate models freeing  $\pi$  or  $\rho$ .

# Parameter estimation

By *Variational EM*, as proposed by Daudin et al., 2008; Chabert-Liddell et al., 2024.

Variational approximation of  $\mathbf{Z}, \mathbf{W} | \mathbf{Y}, \theta^{(t-1)}$

$\mathcal{R}_{Y^m, \tau}(\mathbf{Z}^m, \mathbf{W}^m) = \mathcal{R}_{Y^m, \tau}^1(\mathbf{Z}^m) \times \mathcal{R}_{Y^m, \tau}^2(\mathbf{W}^m) \Rightarrow$  independence rows, columns.

$$\ell(\mathbf{Y}; \theta) \geq \sum_{m=1}^M \left( \mathcal{Q}^m(\theta \mid \theta^{(t)}) + \mathcal{H}(\mathcal{R}_{Y^m, \theta^{(t)}}(\mathbf{Z}^m, \mathbf{W}^m)) \right) =: \mathcal{J}(\tau; \theta)$$

where  $\mathcal{Q}^m(\theta \mid \theta^{(t)}) = \mathbb{E}_{\mathbf{Z}^m, \mathbf{W}^m \sim \mathcal{R}_{Y^m, \tau}(\cdot)} [\ell_c(Y^m, \mathbf{Z}^m, \mathbf{W}^m | \theta)]$

## Problem of choosing $(Q_1, Q_2)$

Need to select  $Q_1$  and  $Q_2$ . BIC-Like criterion<sup>2</sup>

$$\begin{aligned}\text{BIC-L}(\mathbf{Y}, Q_1, Q_2) &= \max_{\theta} \mathbb{E}_{\mathcal{R}_{\mathbf{Y}, \hat{\tau}}} [\ell_c(\mathbf{Y}, \mathbf{Z}, \mathbf{W}; \theta)] + \mathcal{H}(\mathcal{R}_{\mathbf{Y}, \hat{\tau}}) - \frac{1}{2} \text{pen}(\theta, Q_1, Q_2) \\ &= \max_{\theta} \mathcal{J}(\mathcal{R}_{\mathbf{Y}, \hat{\tau}}, \theta) - \frac{1}{2} \text{pen}(\theta, Q_1, Q_2)\end{aligned}$$

### Exploration problems

- Exploration of  $\mathbb{N}^2$  costly.
- Sensitivity to initializations.

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<sup>2</sup>ICL + Entropy + penalty

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### Exploration problems

- Exploration of  $\mathbb{N}^2$  costly. → **Greedy approach** and **sliding window**
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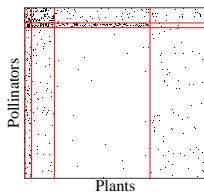
### Exploration problems

- Exploration of  $\mathbb{N}^2$  costly. → **Greedy approach** and **sliding window**
- Sensitivity to initializations. → **Spectral clustering** and **reuse of previous inits**

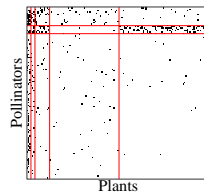
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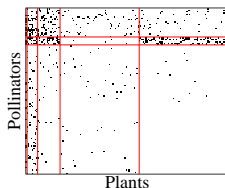
# Results Baldock et al., 2019



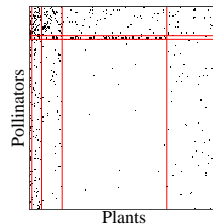
(a) Bristol



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Figure 6: Reordered adjacency matrices by *iid*-colBiSBM, Baldock et al., 2019



# Bombus



(a) *Bombus hortorum* or garden bumblebee



(b) *Bombus lapidarius* or red-tailed bumblebee

# Bombus

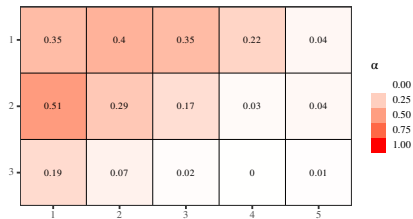


Figure 8: Shared structure of the 4 networks

# Bombus



(a) *Bombus hortorum*  
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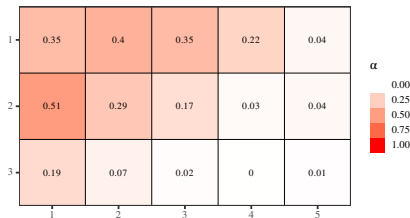


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# Bombus



(b) *Bombus*  
*Lapidarius* or  
red-tailed bumblebee

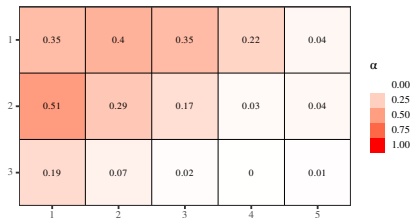


Figure 8: Shared structure of the 4 networks

# Network clustering

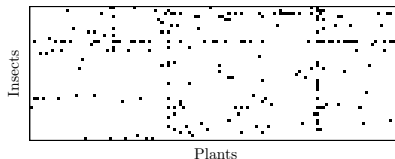


Figure 9: Adjacency matrix, Baldock et al., 2011

# Application to Baldock et al., 2019, 2011 I

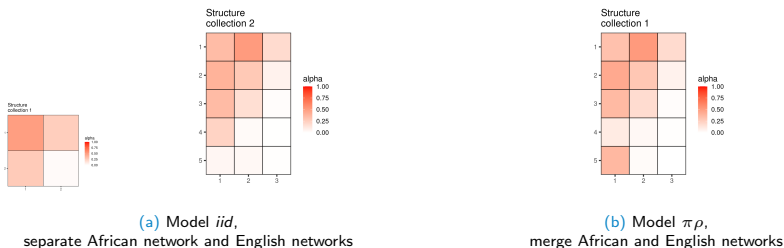
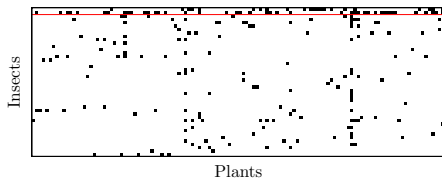
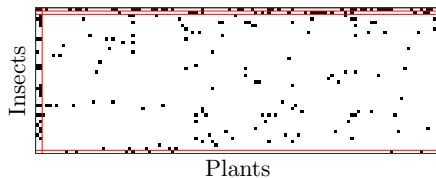


Figure 10: Structures detected for networks of Baldock et al., 2019, 2011

# Results



(a) Reordered by LBM



(b) Reordered by  $\pi\rho$ -colBiSBM

Figure 11: Reordered adjacency matrix by  $\pi\rho$ -colBiSBM, Baldock et al., 2011

# Conclusion and perspectives

## Capabilities

- 4 models including 3 with flexibility on at least one of the dimensions (adaptability to data).
- Detect classic and less classic structures in an agnostic way.
- Partition a set of networks according to their structures.



# Perspectives

- Investigate stability against randomness and local *optima*.

## Package and applications

- Integration into the colSBM package, improvement of user interface and addition of ecologists' feedback
- CRAN submission
- Integrate the possibility of an additional criterion for clustering (e.g. urbanization gradient [Fisogni et al., 2022](#))
- Apply clustering to data from [Pichon et al., 2024](#); [Doré et al., 2021](#)

Thank you for your attention !

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Doré, M., Fontaine, C., & Thébault, E. (2021). Relative effects of anthropogenic pressures, climate, and sampling design on the structure of pollination networks at the global scale. *Global Change Biology*, 27(6), 1266–1280. <https://doi.org/10.1111/gcb.15474>

## Developed formula of variational EM

$$\begin{aligned} \ell(\mathbf{Y}; \theta) \geq & \sum_{m=1}^M \left( \sum_{i=1}^{n_1^m} \sum_{j=1}^{n_2^m} \sum_{q \in Q_{1,m}} \sum_{r \in Q_{2,m}} \tau_{i,q}^{1,m} \tau_{j,r}^{2,m} \log f(Y_{ij}^m; \alpha_{qr}) \right. \\ & + \sum_{i=1}^{n_1^m} \sum_{q \in Q_{1,m}} \tau_{i,q}^{1,m} \log \pi_q^m + \sum_{j=1}^{n_2^m} \sum_{r \in Q_{2,m}} \tau_{j,r}^{2,m} \log \rho_r^m \\ & \left. - \sum_{i=1}^{n_1} \tau_{i,q}^{1,m} \log \tau_{i,q}^{1,m} - \sum_{j=1}^{n_2} \tau_{j,r}^{2,m} \log \tau_{j,r}^{2,m} \right) =: \mathcal{J}(\tau; \theta), \end{aligned}$$

### Variational approximation

$$\tau_{iq}^{1,m} = \mathcal{R}_{Y^m, \tau}^1(Z_{iq}^m = 1) \text{ and } \tau_{jr}^{2,m} = \mathcal{R}_{Y^m, \tau}^2(W_{jr}^m = 1)$$

## Variational Expectation Step

$$\hat{\tau}^{(t+1)} = \arg \max_{\tau} \mathcal{J}(\tau, \hat{\theta}^{(t)}) \Leftrightarrow \arg \min_{\tau \in \mathcal{T}} \mathbf{KL}[\mathcal{R}_{\mathbf{Y}, \tau}, \mathbb{P}(.|\mathbf{Y})]$$

$$\begin{cases} \hat{\tau}_{iq}^{1,m} \propto \hat{\pi}_q^{m(t)} \prod_{j=1}^{n_2^m} \prod_{r \in \mathcal{Q}_2^m} f(Y_{ij}^m; \hat{\alpha}_{qr}^{(t)})^{\hat{\tau}_{jr}^{2,m(t+1)}} & \forall i = 1, \dots, n_1^m, q \in \mathcal{Q}_1^m \\ \hat{\tau}_{jr}^{2,m} \propto \hat{\rho}_r^{m(t)} \prod_{i=1}^{n_1^m} \prod_{q \in \mathcal{Q}_1^m} f(Y_{ij}^m; \hat{\alpha}_{qr}^{(t)})^{\hat{\tau}_{iq}^{1,m(t+1)}} & \forall j = 1, \dots, n_2^m, r \in \mathcal{Q}_2^m \end{cases}$$

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<sup>2</sup>Initialization of  $\hat{\tau}$  with a *spectral clustering* on the networks.

## Maximization Step

$$\hat{\theta}^{(t+1)} = \arg \max_{\theta} \mathcal{J}(\hat{\tau}^{(t+1)}, \theta)$$

### Connectivity parameters

$$\hat{\alpha}_{qr} = \frac{\sum_{m=1}^M \sum_{i=1}^{n_1^m} \sum_{j=1}^{n_2^m} \tau_{iq}^{1,m} \tau_{jr}^{2,m} Y_{ij}^m}{\sum_{m=1}^M \sum_{i=1}^{n_1^m} \sum_{j=1}^{n_2^m} \tau_{iq}^{1,m} \tau_{jr}^{2,m}}$$

### Proportions for *iid*

$$\hat{\pi}_q = \frac{\sum_{m=1}^M \sum_{i=1}^{n_1^m} \tau_{iq}^{1,m}}{\sum_{m=1}^M n_1^m}$$

$$\hat{\rho}_r = \frac{\sum_{m=1}^M \sum_{j=1}^{n_2^m} \tau_{jr}^{2,m}}{\sum_{m=1}^M n_2^m}$$



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### Proportions for $\pi\rho$

$$\hat{\pi}^m_q = \frac{\sum_{i=1}^{n_1^m} \tau_{iq}^{1,m}}{n_1^m} \quad \hat{\rho}^m_r = \frac{\sum_{j=1}^{n_2^m} \tau_{jr}^{2,m}}{n_2^m}$$

# Why does VE minimizes KL ?

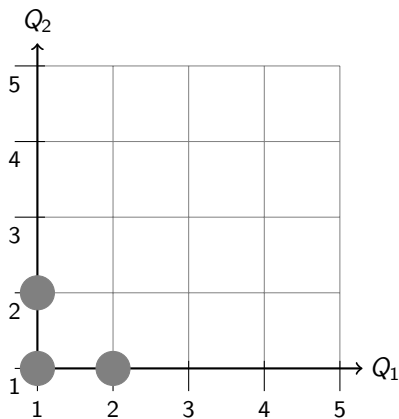
$$\begin{aligned}
 \ell_c(\mathbf{Y}, \mathbf{Z}, \mathbf{W}; \theta) &= \log \mathbb{P}(\mathbf{Z}, \mathbf{W} | \mathbf{Y}; \theta) + \ell(\mathbf{Y}; \theta) \\
 &\Leftrightarrow \ell(\mathbf{Y}; \theta) = \ell_c(\mathbf{Y}, \mathbf{Z}, \mathbf{W}; \theta) - \log \mathbb{P}(\mathbf{Z}, \mathbf{W} | \mathbf{Y}; \theta) \\
 &\Leftrightarrow \mathbb{E}_{\mathcal{R}_{\mathbf{Y}, \tau}}[\ell(\mathbf{Y}; \theta)] = \mathbb{E}_{\mathcal{R}_{\mathbf{Y}, \tau}}[\ell_c(\mathbf{Y}, \mathbf{Z}, \mathbf{W}; \theta)] - \mathbb{E}_{\mathcal{R}_{\mathbf{Y}, \tau}}[\log \mathbb{P}(\mathbf{Z}, \mathbf{W} | \mathbf{Y}; \theta)] \\
 &\Leftrightarrow \ell(\mathbf{Y}; \theta) = \mathbb{E}_{\mathcal{R}_{\mathbf{Y}, \tau}}[\ell_c(\mathbf{Y}, \mathbf{Z}, \mathbf{W}; \theta)] - \mathbb{E}_{\mathcal{R}_{\mathbf{Y}, \tau}}[\log \mathbb{P}(\mathbf{Z}, \mathbf{W} | \mathbf{Y}; \theta)]
 \end{aligned}$$

$$\begin{aligned}
 \text{But } \mathbf{KL}[\mathcal{R}_{\mathbf{Y}, \tau}, \log \mathbb{P}(\mathbf{Z}, \mathbf{W} | \mathbf{Y}; \theta)] &= -\mathbb{E}_{\mathcal{R}_{\mathbf{Y}, \tau}}\left[\log \frac{\mathbb{P}(\mathbf{Z}, \mathbf{W} | \mathbf{Y}; \theta)}{\mathcal{R}_{\mathbf{Y}, \tau}}\right] \\
 &= -\mathbb{E}_{\mathcal{R}_{\mathbf{Y}, \tau}}[\log \mathbb{P}(\mathbf{Z}, \mathbf{W} | \mathbf{Y}; \theta)] + \underbrace{\mathbb{E}_{\mathcal{R}_{\mathbf{Y}, \tau}}[\log \mathcal{R}_{\mathbf{Y}, \tau}]}_{-\mathcal{H}(\mathcal{R}_{\mathbf{Y}, \tau})}
 \end{aligned}$$

$$\Leftrightarrow \mathbf{KL}[\mathcal{R}_{\mathbf{Y}, \tau}, \log \mathbb{P}(\mathbf{Z}, \mathbf{W} | \mathbf{Y}; \theta)] + \mathcal{H}(\mathcal{R}_{\mathbf{Y}, \tau}) = -\mathbb{E}_{\mathcal{R}_{\mathbf{Y}, \tau}}[\log \mathbb{P}(\mathbf{Z}, \mathbf{W} | \mathbf{Y}; \theta)]$$

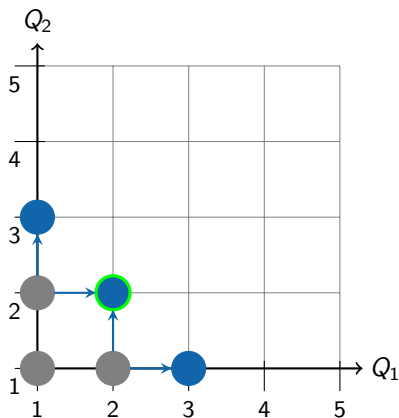
$$\text{Thus } \ell(\mathbf{Y}; \theta) - \mathbf{KL}[\mathcal{R}_{\mathbf{Y}, \tau}, \log \mathbb{P}(\mathbf{Z}, \mathbf{W} | \mathbf{Y}; \theta)] = \mathcal{J}(\tau; \theta) \quad \square$$




# Choice of $(Q_1, Q_2)$ - Greedy approach



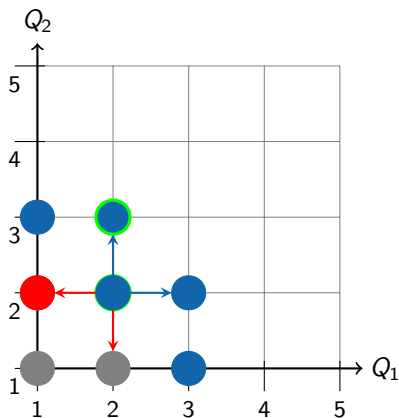
Initial model :





# Choice of $(Q_1, Q_2)$ - Greedy approach



- Initial model : 
- Model after *split* : 
- Model maximizing the criterion : 

# Choice of $(Q_1, Q_2)$ - Greedy approach



- Initial model : 
- Model after *split* : 
- Model maximizing the criterion : 
- Model after *merge* : 

# Choice of $(Q_1, Q_2)$ - Sliding window

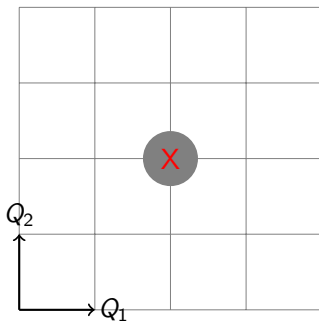


Figure 12: Sliding window

## Choice of $(Q_1, Q_2)$ - Sliding window

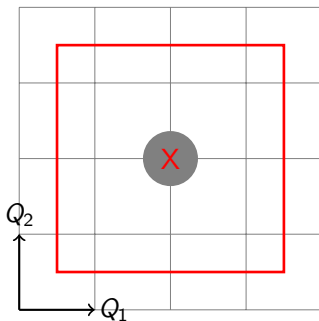


Figure 12: Sliding window

## Choice of $(Q_1, Q_2)$ - Sliding window

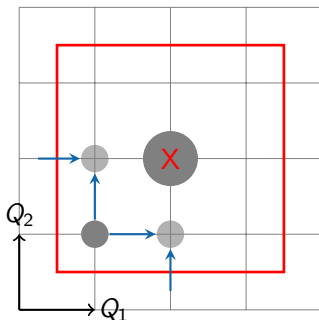


Figure 12: Sliding window

Initialization of the model if necessary



## Choice of $(Q_1, Q_2)$ - Sliding window

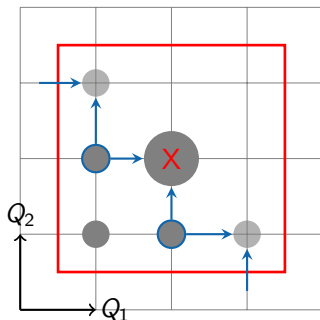


Figure 12: Sliding window

# Choice of $(Q_1, Q_2)$ - Sliding window

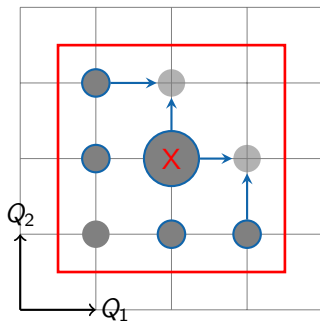


Figure 12: Sliding window

## Choice of $(Q_1, Q_2)$ - Sliding window

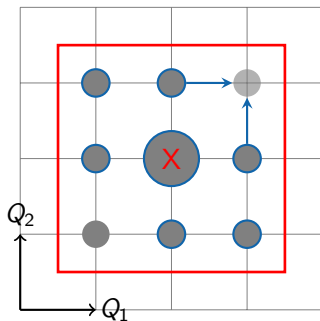


Figure 12: Sliding window

# Choice of $(Q_1, Q_2)$ - Sliding window

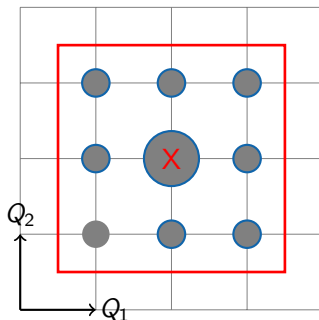


Figure 12: Sliding window

## Choice of $(Q_1, Q_2)$ - Sliding window

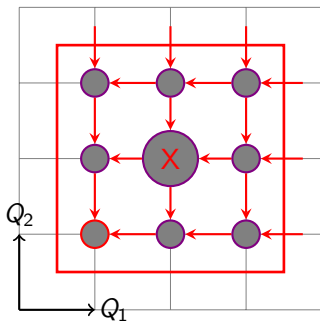


Figure 12: Sliding window

## Choice of $(Q_1, Q_2)$ - Sliding window

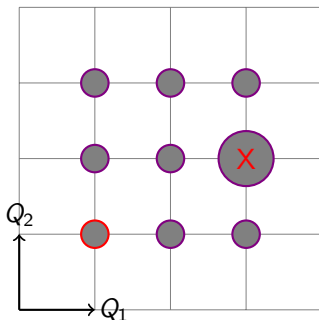


Figure 12: Sliding window

Localization of the new mode

## Choice of $(Q_1, Q_2)$ - Sliding window

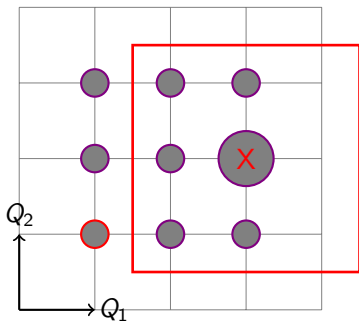
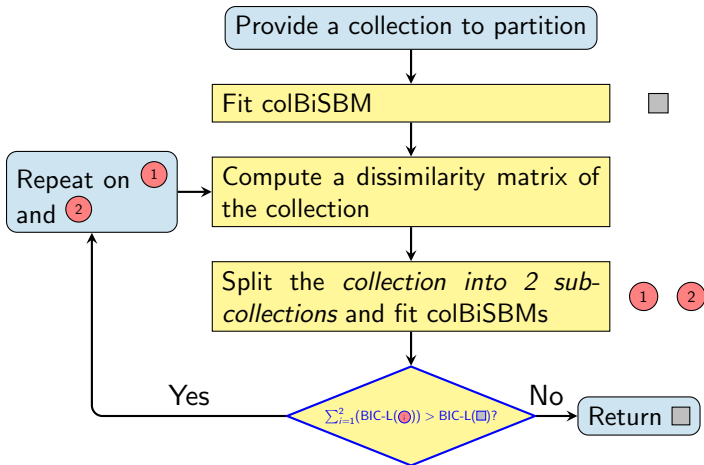


Figure 12: Sliding window

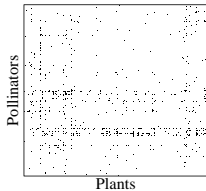
Move to the new mode then iterate

# Clustering algorithm

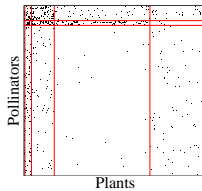


$$D_{\mathcal{M}}(m, m') = \sum_{q=1}^{Q_1} \sum_{r=1}^{Q_2} \max(\tilde{\pi}_q^m, \tilde{\pi}_q^{m'}) \left( \tilde{\alpha}_{qr}^m - \tilde{\alpha}_{qr}^{m'} \right)^2 \max(\tilde{\rho}_r^m, \tilde{\rho}_r^{m'})$$



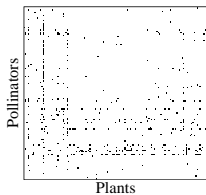


(a) Donnée

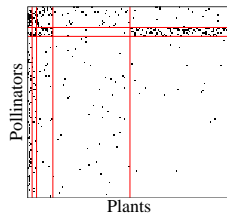


(b) Reordered

Figure 13: Bristol

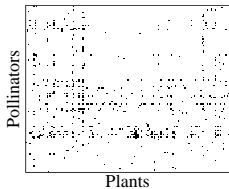


(a) Donnée

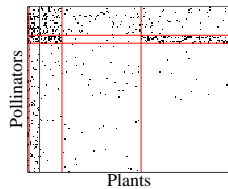


(b) Reordered

Figure 14: Edinburgh

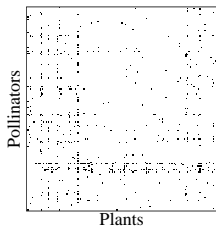


(a) Donnée

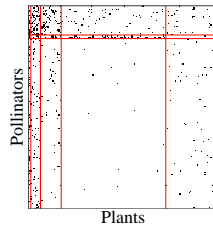


(b) Réordonnée

Figure 15: Leeds



(a) Donnée



(b) Réordonnée

Figure 16: Reading

## Appendices references I