Joint analysis of bipartite networks collection

JdS 2025

Louis Lacoste, Pierre Barbillon and Sophie Donnet
Laboratoire MIA Paris-Saclay

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Why a network?

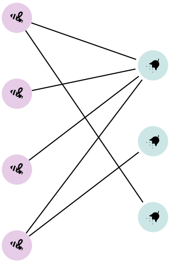

Figure 1: Example of a network

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Associated bi-adjacency matrix

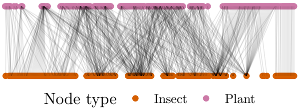

Figure 2: Plant-pollinator network from Bristol Baldock et al., 2019

- Increasingly available
- Modeling of various interactions, here ecosystems
- Structure necessary for: biodiversity monitoring, robustness, risk of collapse

Analysis methods for a network

Several methods :

- Metrics at
  - node level: degree, centrality...
  - network level: density, nestedness...

  Kolaczyk, 2009
- Node embedding and/or clustering with latent variable models
  Snijders and Nowicki, 1997; Hoff et al., 2002
- Node or network embedding with Graph Convolutional Networks
  Kipf and Welling, 2016

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Bipartite Stochastic Block Model (BiSBM[1])

Govaert and Nadif, 2005

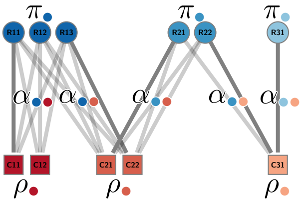


Figure 3: Example of LBM[1]

Hierarchical model

$$\forall q \in [\![1, Q_1]\!],\ \mathbb{P}(Z_i = q) = \pi_q$$
$$\forall r \in [\![1, Q_2]\!],\ \mathbb{P}(W_j = r) = \rho_r$$
$$Y_{ij} | Z_i, W_j \sim \mathcal{F}(\alpha_{Z_i, W_j})$$

where

$$|\pi| = Q_1, |\rho| = Q_2, |\alpha| = Q_1 \times Q_2$$

Concise LBM formula

$$Y \sim \mathcal{F}\text{-BiSBM}_{n_1, n_2}(Q_1, Q_2, \pi, \rho, \alpha)$$

[1]Commonly Known as *Latent Block Model* (LBM) in the literature.

Bipartite Stochastic Block Model (BiSBM[1])

Govaert and Nadif, 2005

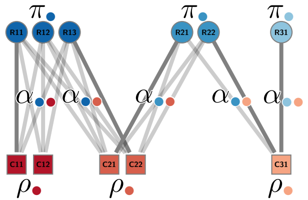


Figure 3: Example of LBM[1]

With

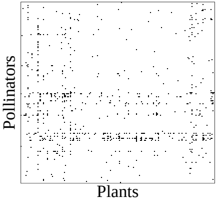
- $Q_1 = |\{\bullet, \bullet, \bullet\}|$ fixed row blocks
- $Q_2 = |\{\bullet, \bullet, \bullet\}|$ fixed column blocks

Parameters

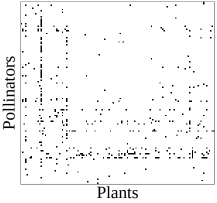
- $\pi_\bullet = \mathbb{P}(Z_i = \bullet)$
- $\rho_\bullet = \mathbb{P}(W_j = \bullet)$
- $\alpha_{\bullet\bullet} = \mathbb{P}(Y_{ij} = 1 | Z_i = \bullet, W_j = \bullet)$

[1] Commonly Known as *Latent Block Model* (LBM) in the literature.

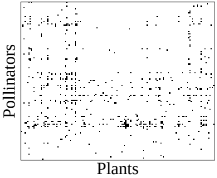
Multiple networks



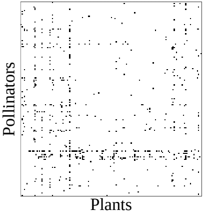
(a) Bristol



(b) Edinburgh



(c) Leeds



(d) Reading

Figure 4: Adjacency matrices, Baldock et al., 2019

Multiple networks

Figure 5: Map of the four cities

Model 0: sep-BiSBM

$$\forall m \in \{1 \dots M\}, Y^m \overset{ind}{\sim} \mathcal{F}\text{-BiSBM}_{n_1^m, n_2^m}({Q_1}^m, {Q_2}^m, \pi^m, \rho^m, \alpha^m)$$

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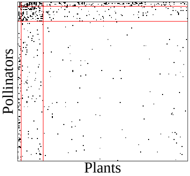

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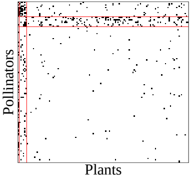

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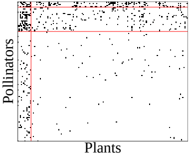

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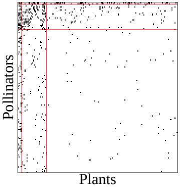

(d) Reading

Figure 6: Reordered adjacency matrices, using BiSBM for each network

Several joint models

iid-colBiSBM

$$\forall m \in \{1 \dots M\}, Y^m \overset{iid}{\sim} \mathcal{F}\text{-BiSBM}_{n_1^m, n_2^m}(Q_1, Q_2, \pi, \rho, \alpha)$$

with $\theta = (\pi, \rho, \alpha)$.

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$\pi\rho$-colBiSBM

$$\forall m \in \{1 \dots M\}, Y^m \overset{ind}{\sim} \mathcal{F}\text{-BiSBM}_{n_1^m, n_2^m}(Q_1, Q_2, \pi^m, \rho^m, \alpha)$$

with $\theta = ((\pi^{m})_{m=1,\dots,M}, (\rho^{m})_{m=1,\dots,M}, \alpha)$.

And intermediate models freeing π or ρ.

Parameter estimation

By *Variational EM*, as proposed by Daudin et al., 2008; Chabert-Liddell et al., 2024.

Variational approximation of $\mathbf{Z}, \mathbf{W} | \mathbf{Y}, \theta^{(t-1)}$

$\mathcal{R}_{Y^m,\tau}(Z^m, W^m) = \mathcal{R}^1_{Y^m,\tau}(Z^m) \times \mathcal{R}^2_{Y^m,\tau}(W^m) \Rightarrow$ independence between rows and columns.

$$\ell(\mathbf{Y};\theta) \geq \sum_{m=1}^{M} \left(\mathcal{Q}^m(\theta \mid \theta^{(t)}) + \mathcal{H}(\mathcal{R}_{Y^m,\theta^{(t)}}(Z^m, W^m)) \right) =: \mathcal{J}(\tau;\theta)$$

where $\mathcal{Q}^m(\theta \mid \theta^{(t)}) = \mathbb{E}_{Z^m,W^m \sim \mathcal{R}_{Y^m,\tau}(.)} \left[\ell_c(Y^m, Z^m, W^m|\theta)\right]$

Problem of choosing (Q_1, Q_2)

Need to select Q_1 and Q_2. BIC-Like criterion[2]

$$\begin{aligned}\text{BIC-L}(\mathbf{Y}, Q_1, Q_2) &= \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\hat{\tau}}}[\ell_c(\mathbf{Y}, \mathbf{Z}, \mathbf{W}; \hat{\theta}^{\text{var}})] + \mathcal{H}(\mathcal{R}_{\mathbf{Y},\hat{\tau}}) - \frac{1}{2}\text{pen}(Q_1, Q_2) \\ &= \mathcal{J}(\mathcal{R}_{\mathbf{Y},\hat{\tau}}, \hat{\theta}^{\text{var}}) - \frac{1}{2}\text{pen}(Q_1, Q_2)\end{aligned}$$

Exploration problems

- Exploration of a 2D grid is costly.
- Sensitivity to initializations.

[2] ICL + entropy - penalty

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- Exploration of a 2D grid is costly. $\rightarrow$ **Greedy approach** and **sliding window**
- Sensitivity to initializations.

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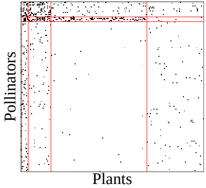
&= \mathcal{J}(\mathcal{R}_{\mathbf{Y},\hat{\tau}}, \hat{\theta}^{\text{var}}) - \frac{1}{2}\text{pen}(Q_1, Q_2)
\end{aligned}
$$

Exploration problems

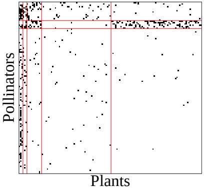
- Exploration of a 2D grid is costly. $\rightarrow$ **Greedy approach** and **sliding window**
- Sensitivity to initializations. $\rightarrow$ **Spectral clustering** and **split & merge** approach

[2] ICL + entropy - penalty

Results Baldock et al., 2019



(a) Bristol



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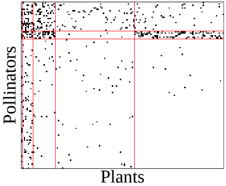
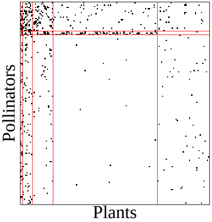


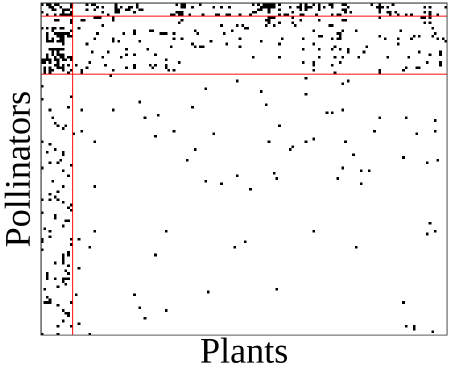
(c) Leeds



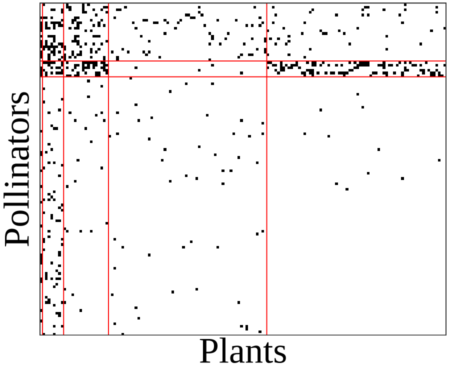
(d) Reading

Figure 7: Reordered adjacency matrices by *iid*-colBiSBM, Baldock et al., 2019

Results Baldock et al., 2019 focus on Leeds



(a) Leeds with sep-BiSBM



(b) Leeds with *iid*-colBiSBM

Bombus

(a) *Bombus Hortorum* or garden bumblebee

(b) *Bombus Lapidarius* or red-tailed bumblebee

Bombus

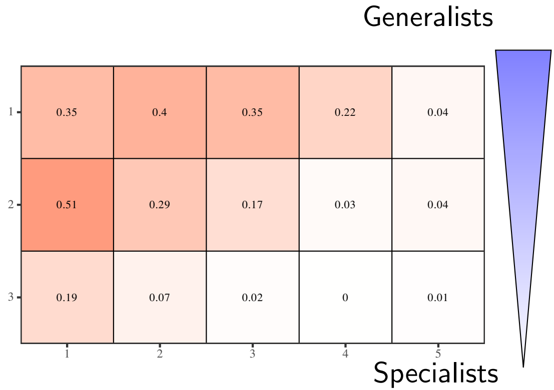


Figure 10: Shared structure (α matrix) of the four networks

Bombus

(a) *Bombus Hortorum* or garden bumblebee

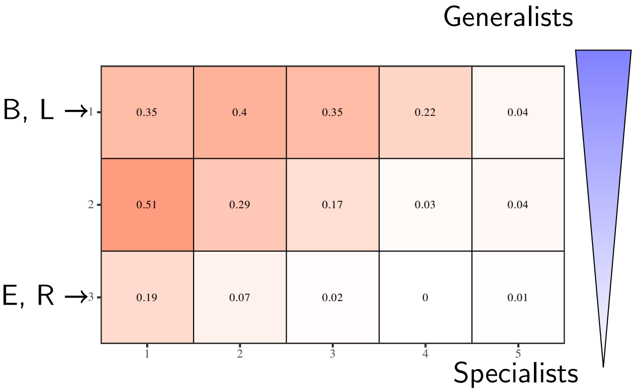


Figure 10: Shared structure (α matrix) of the four networks

Bombus

(b) *Bombus Lapidarius* or red-tailed bumblebee

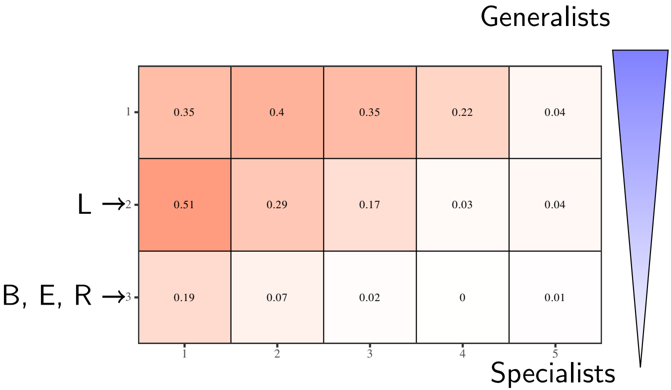
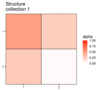


Figure 10: Shared structure (α matrix) of the four networks

Application to Baldock et al., 2019, 2011 I

TODO Put α plots and tree structure of partition



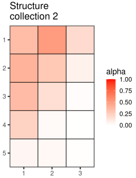


Figure 11: Model *iid*, separate Kenyan (left) and British (right) networks

Conclusion and perspectives

Capabilities

- 4 models including 3 with flexibility on at least one of the dimensions (adaptability to data).
- Detect classic and less classic structures in an agnostic way.
- Partition a set of networks according to their structures.

Package and applications

- `ArXiv` preprint in redaction
- `CRAN` submission
- Integrate the possibility of an additional criterion for clustering (e.g. urbanization gradient Fisogni et al., 2022)
- Apply clustering to data from Pichon et al., 2024; Doré et al., 2021

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Developed formula of variational EM

$$
\begin{aligned}
\ell(\mathbf{Y};\theta) \geq \sum_{m=1}^{M} \Bigg(& \sum_{i=1}^{n_1^m} \sum_{j=1}^{n_2^m} \sum_{q \in \mathcal{Q}_{1,m}} \sum_{r \in \mathcal{Q}_{2,m}} \tau_{i,q}^{1,m} \tau_{j,r}^{2,m} \log f(Y_{ij}^m; \alpha_{qr}) \\
& + \sum_{i=1}^{n_1^m} \sum_{q \in \mathcal{Q}_{1,m}} \tau_{i,q}^{1,m} \log \pi_q{}^m + \sum_{j=1}^{n_2^m} \sum_{r \in \mathcal{Q}_{2,m}} \tau_{j,r}^{2,m} \log \rho_r{}^m \\
& - \sum_{i=1}^{n_1} \tau_{i,q}^{1,m} \log \tau_{i,q}^{1,m} - \sum_{j=1}^{n_2} \tau_{j,r}^{2,m} \log \tau_{j,r}^{2,m} \Bigg) =: \mathcal{J}(\tau;\theta),
\end{aligned}
$$

Variational approximation

$\tau_{iq}^{1,m} = \mathcal{R}^1_{Y^m,\tau}(Z_{iq}^m = 1)$ and $\tau_{jr}^{2,m} = \mathcal{R}^2_{Y^m,\tau}(W_{jr}^m = 1)$

Variational Expectation Step

$$\widehat{\tau}^{(t+1)} = \arg\max_{\tau} \mathcal{J}(\tau, \widehat{\boldsymbol{\theta}}^{(t)}) \Leftrightarrow \arg\min_{\tau \in \mathcal{T}} \mathbf{KL}[\mathcal{R}_{\mathbf{Y},\tau}, \mathbb{P}(.|\mathbf{Y})]$$

$$\begin{cases} \widehat{\tau}^{1,m}_{iq} \propto \widehat{\pi}^{m(t)}_{q} \prod_{j=1}^{n_2^m} \prod_{r \in \mathcal{Q}_2^m} f(Y^m_{ij}; \widehat{\alpha}^{(t)}_{qr})^{\widehat{\tau}^{2,m(t+1)}_{jr}} & \forall i = 1, \ldots, n_1^m, q \in \mathcal{Q}_1^m \\ \widehat{\tau}^{2,m}_{jr} \propto \widehat{\rho}^{m(t)}_{r} \prod_{i=1}^{n_1^m} \prod_{q \in \mathcal{Q}_1^m} f(Y^m_{ij}; \widehat{\alpha}^{(t)}_{qr})^{\widehat{\tau}^{1,m(t+1)}_{iq}} & \forall j = 1, \ldots, n_2^m, r \in \mathcal{Q}_2^m \end{cases}$$

[2] Initialization of $\widehat{\tau}$ with a *spectral clustering* on the networks.

Maximization Step

$$\widehat{\theta}^{(t+1)} = \arg\max_{\theta} \mathcal{J}(\widehat{\boldsymbol{\tau}}^{(t+1)}, \theta)$$

Connectivity parameters

$$\widehat{\alpha}_{qr} = \frac{\sum_{m=1}^{M} \sum_{i=1}^{n_1^m} \sum_{j=1}^{n_2^m} \tau_{iq}^{1,m} \tau_{jr}^{2,m} Y_{ij}^m}{\sum_{m=1}^{M} \sum_{i=1}^{n_1^m} \sum_{j=1}^{n_2^m} \tau_{iq}^{1,m} \tau_{jr}^{2,m}}$$

Proportions for *iid*

$$\widehat{\pi}_q = \frac{\sum_{m=1}^{M} \sum_{i=1}^{n_1^m} \tau_{iq}^{1,m}}{\sum_{m=1}^{M} n_1^m} \qquad \widehat{\rho}_r = \frac{\sum_{m=1}^{M} \sum_{j=1}^{n_2^m} \tau_{jr}^{2,m}}{\sum_{m=1}^{M} n_2^m}$$

Maximization Step

$$\widehat{\theta}^{(t+1)} = \arg\max_{\theta} \mathcal{J}(\widehat{\boldsymbol{\tau}}^{(t+1)}, \theta)$$

Connectivity parameters

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Proportions for $\pi \rho$

$$\widehat{\pi}^{m}{}_{q} = \frac{\sum_{i=1}^{n_1^m} \tau_{iq}^{1,m}}{n_1^m} \qquad \widehat{\rho}^{m}{}_{r} = \frac{\sum_{j=1}^{n_2^m} \tau_{jr}^{2,m}}{n_2^m}$$

Why does VE minimizes KL ?

$$\begin{aligned}
\ell_c(\mathbf{Y},\mathbf{Z},\mathbf{W};\theta) &= \log\mathbb{P}(\mathbf{Z},\mathbf{W}|\mathbf{Y};\theta) + \ell(\mathbf{Y};\theta)\\
\Leftrightarrow \ell(\mathbf{Y};\theta) &= \ell_c(\mathbf{Y},\mathbf{Z},\mathbf{W};\theta) - \log\mathbb{P}(\mathbf{Z},\mathbf{W}|\mathbf{Y};\theta)\\
\Leftrightarrow \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\ell(\mathbf{Y};\theta)] &= \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\ell_c(\mathbf{Y},\mathbf{Z},\mathbf{W};\theta)] - \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log\mathbb{P}(\mathbf{Z},\mathbf{W}|\mathbf{Y};\theta)]\\
\Leftrightarrow \ell(\mathbf{Y};\theta) &= \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\ell_c(\mathbf{Y},\mathbf{Z},\mathbf{W};\theta)] - \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log\mathbb{P}(\mathbf{Z},\mathbf{W}|\mathbf{Y};\theta)]
\end{aligned}$$

$$\text{But } \mathbf{KL}[\mathcal{R}_{\mathbf{Y},\tau},\log\mathbb{P}(\mathbf{Z},\mathbf{W}|\mathbf{Y};\theta)] = -\mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log\frac{\mathbb{P}(\mathbf{Z},\mathbf{W}|\mathbf{Y};\theta)}{\mathcal{R}_{\mathbf{Y},\tau}}]$$

$$= -\mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log\mathbb{P}(\mathbf{Z},\mathbf{W}|\mathbf{Y};\theta)] + \underbrace{\mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}[\log\mathcal{R}_{\mathbf{Y},\tau}]}}_{-\mathcal{H}(\mathcal{R}_{\mathbf{Y},\tau})}$$

$$\Leftrightarrow \mathbf{KL}[\mathcal{R}_{\mathbf{Y},\tau},\log\mathbb{P}(\mathbf{Z},\mathbf{W}|\mathbf{Y};\theta)] + \mathcal{H}(\mathcal{R}_{\mathbf{Y},\tau}) = -\mathbb{E}_{\mathcal{R}_{\mathbf{Y},\tau}}[\log\mathbb{P}(\mathbf{Z},\mathbf{W}|\mathbf{Y};\theta)]$$

Thus $\ell(\mathbf{Y};\theta) - \mathbf{KL}[\mathcal{R}_{\mathbf{Y},\tau},\log\mathbb{P}(\mathbf{Z},\mathbf{W}|\mathbf{Y};\theta)] = \mathcal{J}(\tau;\theta)$ □

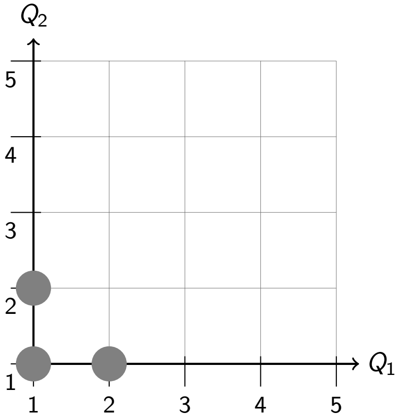
On the BIC-L

Raconter l'histoire dans l'ordre suivant :

- ICL = Méthode BIC (approx Laplace) sur la log complète, fait apparaître la pénalité de complexité et pénalise l'entropie
- ICLv = ICL mais avec les paramètres variationnels et l'entropie variationnelle
- BIC-L = ICLv mais sans la pénalité sur l'entropie et la rajoutant à la fin

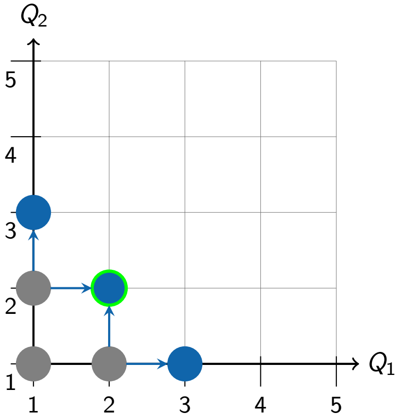
$$\begin{aligned}
\mathrm{BIC}(\hat{\theta}) &= \log p(\mathbf{Y}; \hat{\theta}) - \frac{1}{2}\mathrm{pen}(\dots) \\
&= \mathbb{E}_{\mathbf{Z},\mathbf{W}|\mathbf{Y}}[\underbrace{\log p(\mathbf{Y}, \mathbf{Z}, \mathbf{W}; \hat{\theta})}_{\ell_c(\mathbf{Y},\mathbf{Z},\mathbf{W};\hat{\theta})}] + \mathcal{H}(p(\mathbf{Z}, \mathbf{W}|\mathbf{Y})) - \frac{1}{2}\mathrm{pen}(\dots) \\
\mathrm{ICL}(\hat{\theta}) &= \mathbb{E}_{\mathbf{Z},\mathbf{W}|\mathbf{Y}}[\ell_c(\mathbf{Y}, \mathbf{Z}, \mathbf{W}; \hat{\theta})] - \frac{1}{2}\mathrm{pen}(\dots) \\
\mathrm{BIC\text{-}L}(\hat{\theta}, \hat{\tau}) &= \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\hat{\tau}}}[\ell_c(\mathbf{Y}, \mathbf{Z}, \mathbf{W}; \hat{\theta}^{\mathrm{var}})] + \mathcal{H}(\mathcal{R}_{\mathbf{Y},\hat{\tau}}) - \frac{1}{2}\mathrm{pen}(\dots)
\end{aligned}$$

Choice of (Q_1, Q_2) - Greedy approach



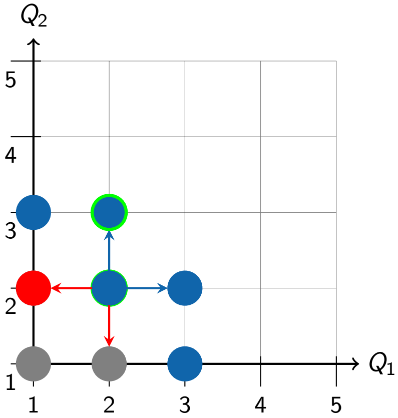
- Initial model :

Choice of (Q_1, Q_2) - Greedy approach



- Initial model :
- Model after *split* :
- Model maximizing the criterion :

Choice of (Q_1, Q_2) - Greedy approach



- Initial model :
- Model after *split* :
- Model maximizing the criterion :
- Model after *merge* :

Choice of (Q_1, Q_2) - Sliding window

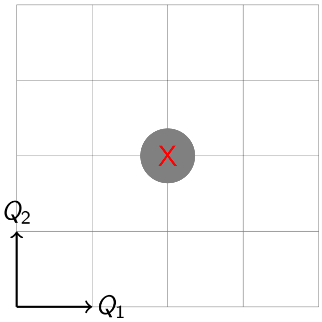


Figure 12: Sliding window

Choice of (Q_1, Q_2) - Sliding window

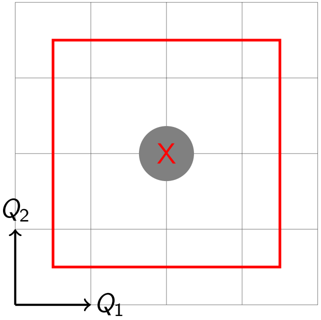


Figure 12: Sliding window

Choice of (Q_1, Q_2) - Sliding window

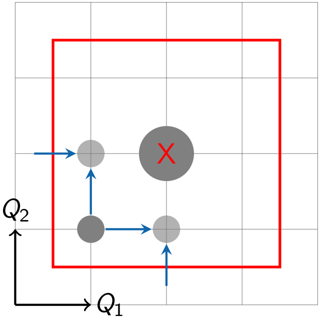


Figure 12: Sliding window

Initialization of the model if necessary

Choice of (Q_1, Q_2) - Sliding window

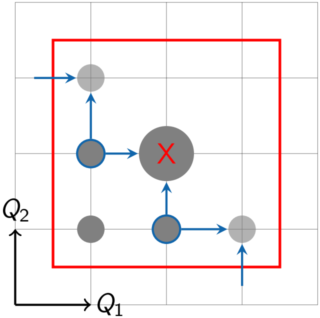


Figure 12: Sliding window

Choice of (Q_1, Q_2) - Sliding window

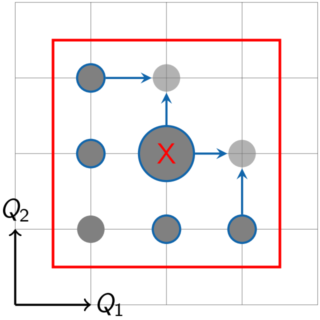


Figure 12: Sliding window

Choice of (Q_1, Q_2) - Sliding window

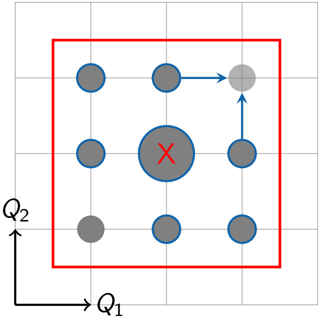


Figure 12: Sliding window

Choice of (Q_1, Q_2) - Sliding window

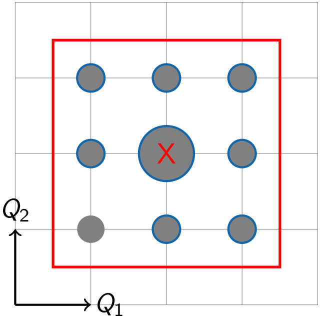


Figure 12: Sliding window

Choice of (Q_1, Q_2) - Sliding window

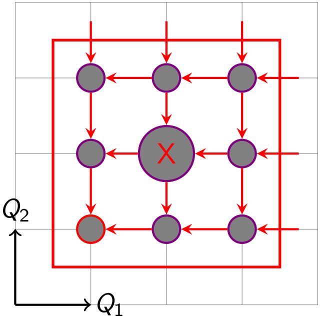


Figure 12: Sliding window

Choice of (Q_1, Q_2) - Sliding window

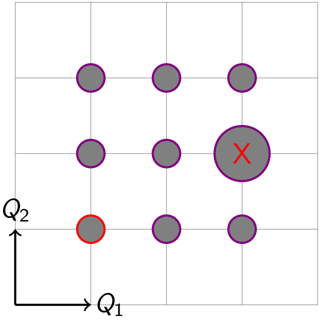


Figure 12: Sliding window

Localization of the new mode

Choice of (Q_1, Q_2) - Sliding window

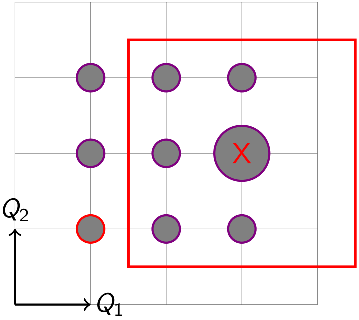
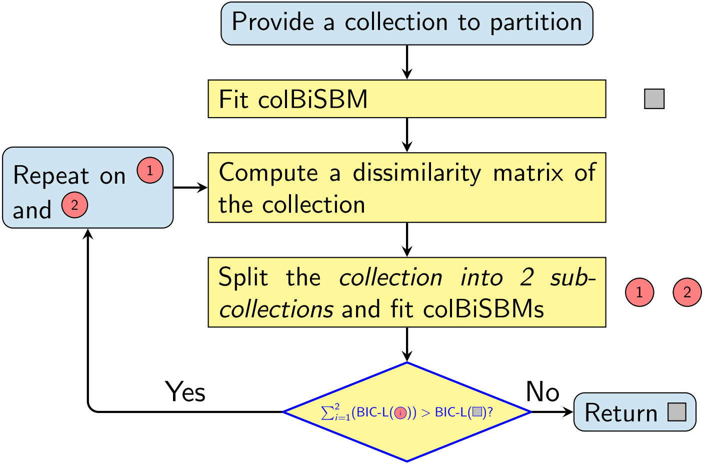


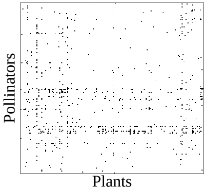
Figure 12: Sliding window

Move to the new mode then iterate

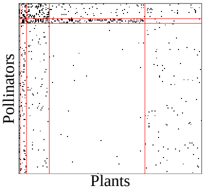
Clustering algorithm



$$D_{\mathcal{M}}(m, m') = \sum_{q=1}^{Q_1} \sum_{r=1}^{Q_2} \max(\widetilde{\pi}_q^{m}, \widetilde{\pi}_q^{m'}) \left(\widetilde{\alpha}_{qr}^{m} - \widetilde{\alpha}_{qr}^{m'} \right)^2 \max(\widetilde{\rho}_r^{m}, \widetilde{\rho}_r^{m'})$$

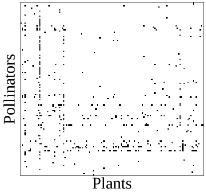


(a) Donnée

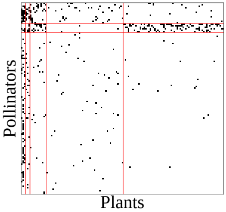


(b) Reordered

Figure 13: Bristol



(a) Donnée



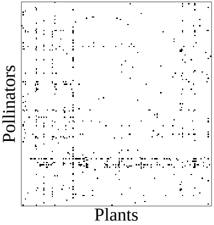
(b) Reordered

Figure 14: Edinburgh

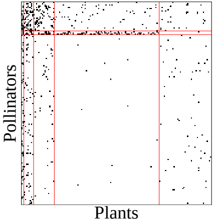
(a) Donnée

(b) Réordonnée

Figure 15: Leeds



(a) Donnée



(b) Réordonnée

Figure 16: Reading

Appendices references I